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A GENERALIZATION OF A THEOREM BY BOSWELL AND PATIL.(U)
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A Generalization of a Theorem

by Boswell and Patil

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SUMMARY

The class of convoluted Pascal distributions obeying certain moment conditions are characterized by a system of differential equations satisfied by their probability mass functions. The result contains a characterization of Pascal distributions obtained by Boswell and Patil (1973) as a special case.

I. INTRODUCTION

Boswell and Patil (1973) characterized several families of discrete distributions by differential equations satisfied by their probability mass functions. Among the families they studied were Poisson distributions, Pascal distributions and Binomial distributions. Samaniego (1976) obtained a characterization of convoluted Poisson distributions which generalizes their work on the Poisson distribution. It is our purpose here to extend their characterization of Pascal distributions to discrete convolutions of Pascal distributions. Characterizations in terms of differentiation with respect to appropriate parameters have been found useful in the calculation of expectations (see, for example, Price (1958)), and have also proved useful in point and interval estimation (see Samaniego (1976, 1977)).

II. THE CHARACTERIZATION THEOREM

The family of Pascal distributions is a two-parameter family of distributions on the nonnegative integers having probability mass functions

$$f(x|n,\theta) = \binom{n+x-1}{x}(\theta-1)^x/\theta^{x+n}, \quad x=0,1,\cdots, \tag{1}$$

where $n=1,2,\cdots$ and $\theta \geq 1$. This distribution is often referred to as the negative binomial distribution, although the usual parametrization is in terms of the parameter $p = 1/\theta$. The following result is due to Boswell and Patil (1973).

Theorem 1. Let $\{X_{n,\theta}\}$ be a family of nonnegative integer valued random variables indexed by parameters $\theta \geq 1$, $n=1,2,\cdots$, and let $X_{0,\theta} \equiv X_{n,1} \equiv 0$. Furthermore, assume moments of all orders exist for every n and θ. Then

$$\frac{\partial}{\partial\theta} f(x|n,\theta) = n[f(x-1|n+1,\theta) - f(x|n+1,\theta)] \quad \forall n,\ \forall\theta$$

if and only if $\underset{\sim}{X}(n,\theta)$ has a Pascal distribution with probability mass function (1).

In generalizing this result to convoluted Pascal distributions, we make use of several lemmas. The first, due to Carleman, is proven in Breiman (1968).

Lemma 1. Let $\{\mu_k\}$ be a sequence of real numbers satisfying

$$\overline{\lim_k} \frac{|\mu_k|^{1/k}}{k} < \infty. \tag{2}$$

Then there is at most one distribution function F satisfying

$$\mu_k = \int x^k F(dx).$$

Lemma 2. Let Y, Z be independent nonnegative random variables with moment sequences $\{\mu_k\}$ and $\{\nu_k\}$, respectively, satisfying condition (2). If X = Y + Z, then the moment sequence of X satisfies condition (2).

Proof. Let $\lambda_k = EX^k$, $k=1,2,\cdots$. We may write

$$\lambda_k = \sum_{i=0}^{k} \binom{k}{i} \mu_i \nu_{k-i}$$

$$\leq \sum_{i=0}^{k} \binom{k}{i} \mu_k^{i/k} \nu_k^{(k-i)/k}$$

since absolute moments $\{\beta_i\}$ satisfy $\beta_r^{1/r} \leq \beta_{r+1}^{1/(r+1)}$ (see Lukacs (1970), p. 14). Thus,

$$\lambda_k \leq (\mu_k^{1/k} + \nu_k^{1/k})^k$$

and

$$\lambda_k^{1/k} \leq \mu_k^{1/k} + \nu_k^{1/k} .$$

which implies $\overline{\lim}\ \lambda_k^{1/k} < \infty$.

Lemma 3. The Pascal distribution has a moment sequence satisfying condition (2).

Proof. It is well known, of course, that the Pascal distribution is determined by its moments. Condition (2) is sufficient but not necessary for such uniqueness, and thus we verify that it is indeed satisfied here.

Since the Pascal distribution may be viewed as the distribution of the sum of independent geometric random variables, it suffices by Lemma 2 to show that the geometric distribution has a moment sequence satisfying (2). Let the geometric distribution be parametrized in terms of $p = 1/\theta$ so that its probability mass function is

$$f(x|p) = pq^x \qquad x=0,1,2,\cdots$$

Let $\phi(t)$ be the moment generating function of a geometric variable, i.e.,

$$\phi(t) = p(1-qe^t)^{-1}.$$

One can easily show inductively that $\phi^{(k)}(t) = \frac{\partial^k}{\partial t^k}\phi(t)$ may be written as

$$\phi^{(k)}(t) = \sum_{j=1}^{k} a_j^{(k)} pq^j e^{jt}(1-qe^t)^{-(j+1)}$$

where $a_1^{(k)} = 1$ and $1 < a_j^{(k)} < k^k$ for $j=2,\ldots,k$. The k^{th} moment of the geometric distribution is thus given by

$$\mu_k = \sum_{j=1}^{k} a_j^{(k)} (\frac{q}{p})^j .$$

We thus have

$$\left(\frac{\mu_k^{1/k}}{k}\right)^k = \frac{\mu_k}{k^k}$$

$$= \sum_{j=1}^{k} \frac{a_j^{(k)}}{k^k} (\frac{q}{p})^j$$

$$< \sum_{j=1}^{k} (\frac{q}{p})^j .$$

If $p > 1/2$, then

$$\left(\frac{\mu_k^{1/k}}{k}\right)^k < \sum_{j=1}^{k} \left(\frac{q}{p}\right)^j$$

and

$$\overline{\lim_k} \frac{\mu_k^{1/k}}{k} < \lim_{k \to \infty} \left[\sum_{j=1}^{k} \left(\frac{q}{p}\right)^j\right]^{1/k} = \lim_{k \to \infty} \left(\frac{q}{p-q}\right)^{1/k} \left(1-\left(\frac{q}{p}\right)^k\right)^{1/k} = 1.$$

If $p \leq 1/2$,

$$\frac{\mu_k^{1/k}}{k} \leq \left[\sum_{j=1}^{k} \left(\frac{q}{p}\right)^j\right]^{1/k} \leq \left[k\left(\frac{q}{p}\right)^k\right]^{1/k} = k^{1/k} \frac{q}{p} .$$

Thus,

$$\overline{\lim} \frac{\mu_k^{1/k}}{k} \leq \lim_{k \to \infty} k^{1/k}\left(\frac{q}{p}\right) = \frac{q}{p} .$$

In either case, we have

$$\overline{\lim} \frac{\mu_k^{1/k}}{k} < \infty$$

which is condition (2).

The k^{th} factorial moment of a random variable X will be denoted by $EX_{(k)}$ and represents the expectation $EX(X-1)\cdots(X-k+1)$. The following two lemmas are easily proven inductively.

<u>Lemma 4</u>. Let Y, Z be independent random variables and let $X = Y + Z$. Let $\lambda_{(k)}$, $\mu_{(k)}$ and $\nu_{(k)}$ be the k^{th} factorial moments of X, Y and Z respectively. Then

$$\lambda_{(k)} = \sum_{i=0}^{k} \binom{k}{i} \mu_{(k-i)} \nu_{(i)}.$$

<u>Lemma 5</u>. If Y has a Pascal distribution with parameters n and θ, the k^{th} factorial moment of Y is given by

$$EY_{(k)} = \frac{(n+k-1)!}{(n-1)!} (\theta-1)^k .$$

We are now in a position to prove the following characterization result.

<u>Theorem 2</u>. Let $\{X_{n,\theta}\}$ be a family of nonnegative integer valued random variables indexed by positive integers n and by $\theta > 1$, and let Z be a nonnegative integer valued random variable whose distribution does not depend on θ. Suppose the moment sequence $\{\nu_k\}$ of Z satisfies condition (2), and $EX^k_{n,\theta} \to \nu_k$ as $\theta \to 1$ for all n, k. The probability mass function of $X_{n,\theta}$ satisfies

$$\frac{\partial}{\partial\theta} f(x|n,\theta) = n[f(x-1|n+1,\theta) - f(x|n+1,\theta)] \quad \forall n, \theta \qquad (3)$$

if and only if the distribution of $X_{n,\theta}$ is the convolution of the Pascal distribution with parameters n, θ and the distribution of Z.

Proof. It is easy to verify that the probability mass function of a convoluted Pascal distribution satisfies (3). We proceed immediately to the converse. Let $E(X_{n,\theta})_{(k)}$ represent the k^{th} factorial moment of $X_{n,\theta}$. It is clear that since factorial moments are linear functions of the central moments, $EX_{n,\theta}^k \to EZ^k$ as $\theta \to 1$ implies that $E(X_{n,\theta})_{(k)} \to EZ_{(k)}$ as $\theta \to 1$. Moreover, if a probability distribution is determined by its moments, it is determined by its factorial moments. Let

$$x_{(k)} = x(x-1)\cdots(x-k+1)$$

and note that

$$(x+1)_{(k)} - x_{(k)} = kx_{(k-1)}. \tag{4}$$

If $\{X_{n,\theta}\}$ is a family of random variables whose probability mass functions satisfy the system of differential equations (3), and if moments of all orders exist, as postulated, then from (4) we have for $k \geq 1$

$$\frac{\partial}{\partial\theta} E\left[(X_{n,\theta})_{(k)}\right] = n\left[\sum_{x=1}^{\infty} x_{(k)} f(x-1|n+1,\theta) - \sum_{x=0}^{\infty} x_{(k)} f(x|n+1,\theta)\right]$$

$$= n\left[\sum_{x=0}^{\infty} (x+1)_{(k)} f(x|n+1,\theta) - \sum_{x=0}^{\infty} x_{(k)} f(x|n+1,\theta)\right]$$

$$= nk\, E\left[(X_{n+1,\theta})_{(k-1)}\right] \quad \forall n,\ \theta. \tag{5}$$

We show inductively that the factorial moments of $X_{n,\theta}$ are those of a convoluted Pascal distribution which is determined by its moments. Consider, for fixed n, the first moment of $X_{n,\theta}$. We have from (5)

$$\frac{\partial}{\partial\theta} E(X_{n,\theta}) = n. \tag{6}$$

Equating indefinite integrals of terms in (6) yields

$$E(X_{n,\theta}) = n(\theta-1) + c.$$

Since $\lim_{\theta \to 1} E(X_{n,\theta}) \to EZ$, we have

$$E(X_{n,\theta}) = n(\theta-1) + EZ.$$

Thus, for each n, the first (factorial) moment of $X_{n,\theta}$ is that of $Y_{n,\theta} + Z$ where $Y_{n,\theta}$ has a Pascal distribution with parameters n, θ. Assume that for every n, the k^{th} factorial moment is that of $Y_{n,\theta} + Z$, that is, assume, using Lemmas 4 and 5, that

$$E(X_{n,\theta})_{(k)} = \sum_{i=0}^{k} \binom{k}{i} \frac{(n+i-1)!}{(n-1)!} (\theta-1)^i \nu_{(k-i)}.$$

Consider the (k+1)-st factorial moment. Equation (5) implies that

$$\frac{\partial}{\partial\theta} E(X_{n,\theta})_{(k+1)} = n(k+1) \sum_{i=0}^{k} \binom{k}{i} \frac{(n+i)!}{n!} (\theta-1)^i \nu_{(k-i)}.$$

Thus

$$E(X_{n,\theta})_{(k+1)} = n(k+1) \sum_{i=0}^{k} \binom{k}{i} \frac{(n+i)!}{n!} \frac{(\theta-1)^{i+1}}{i+1} \nu_{(k-i)} + c$$

$$= n(k+1) \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \frac{(n+i)!}{n!} \frac{(\theta-1)^{i+1}}{i+1} \nu_{(k-i)} + c$$

$$= \sum_{i=0}^{k} \binom{k+1}{i+1} \frac{(n+i)!}{(n-1)!} (\theta-1)^{i+1} \nu_{(k-i)} + c$$

$$= \sum_{i=1}^{k+1} \binom{k+1}{i} \frac{(n+i-1)!}{(n-1)!} (\theta-1)^{i} \nu_{(k+1-i)} + c$$

$$= \sum_{i=0}^{k+1} \binom{k+1}{i} \frac{(n+i-1)!}{(n-1)!} (\theta-1)^{i} \nu_{(k+1-i)} + c'.$$

Now since $E(X_{n,\theta})_{(k+1)} \to EZ_{(k+1)}$ as $\theta \to 1$, we may identify c' as zero, and thus the k^{th} factorial moment of $X_{n,\theta}$ is that of $Y_{n,\theta} + Z$ where $Y_{n,\theta}$ has a Pascal distribution with parameters n, θ. By Lemmas 2 and 3, this convoluted Pascal distribution is uniquely determined by its moments, so that the distribution of $X_{n,\theta}$ is the convolution of the Pascal distribution and the distribution of Z.

Remark 1. The characterization theorem for convoluted Pascal distributions reduces to the result obtained by Boswell and Patil when the random variable Z is degenerate.

Remark 2. It is interesting that the system of differential equations (3) are satisfied by any convoluted Pascal distribution, irrespective of the existence of moments. The moment conditions in the hypothesis of Theorem 2 are needed in our proof for unique identification of convoluted Pascal distributions by their moment sequences. We conjecture that the system (3) alone characterizes the class of all convoluted Pascal distributions.

Remark 3. Let $X_1,\ldots,X_n$ be a random sample from a convoluted Pascal distribution. Let the likelihood function be denoted by

$$L(x_1,\ldots,x_k,\theta) = \prod_{i=1}^{k} f(x_i|n,\theta).$$

The following representation of the likelihood equation $\frac{\partial}{\partial\theta} \ln L = 0$ may be useful in maximum likelihood estimation.

$$\frac{\partial}{\partial\theta} \ln L = n \sum_{i=1}^{k} \frac{f(x_i - 1 | n+1, \theta) - f(x_i | n+1, \theta)}{f(x_i | n, \theta)} = 0. \tag{6}$$

REFERENCES

Boswell, M. T. and Patil, G. P. (1973). Characterization of certain discrete distributions by differential equations with respect to their parameters. Australian Journal of Statistics, **15**, 128-131.

Breiman, L. (1968). Probability, Reading, Mass.: Addison-Wesley.

Lukacs, E. (1970). Characteristic Functions, New York: Hafner.

Price, R. (1958). A useful theorem for nonlinear devices having Gaussian inputs. IRE Transactions on Information Theory, IT-4, 69-72.

Samaniego, F. J. (1976). A characterization of convoluted Poisson distributions with applications to estimation. Journal of the American Statistical Association, **71**, 475-479.

Samaniego, F. J. (1977). A characterization of convoluted geometric distributions. Technical Report, Department of Mathematics, University of California, Davis.

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